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ON THE ELASTIC STRIP WITH AN INTERNAL CRACK

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Abstract—The paper presents a method to deal with an inclined crack in an elastic strip. No assumptions of symmetry are made. The method involves the solutions for a cracked plane and an uncracked strip and results in two coupled singular integral equations with finite interval of integration. A crack in a half-plane arises as a limiting case. For internal cracks the integral equations are of a standard type and do not present any numerical difficulties. Results are presented for loads according to the technical beam theory.

1. INTRODUCTION

In recent years stress intensity factors have been determined for a large number of geometries, and many problems relating to cracks in a strip or a half-plane have been treated. A number of these are mentioned below. Some of these problems however have been unnecessarily restricted to symmetrical geometry and loading.

A variety of methods are available. The Wiener–Hopf Technique has been applied in [1, 2] to solve a problem involving edge cracks at right angles to the boundary. The symmetrical problem of an internal transverse crack in a strip was formulated in terms of a dual integral equation and solved in [3]. Later this solution was extended to a cracked strip between two half-planes [4, 5]. The problem of a single edge crack in a strip was recently treated by the same technique in [6]. Most of these problems and several others have also been solved by means of singular integral equations [7–11]. A different formulation also using singular integral equations has been given in [12]. Finally mention must be made of series and collocation techniques described in [13–16].

The purpose of this paper is to treat an oblique crack in an elastic strip under arbitrary loading. The equivalent problem for a cracked half-plane arises as a limiting case. Two coupled singular integral equations with a finite interval of integration are obtained by use of Fourier transforms. These equations remain valid for an edge crack. The numerical solution follows the quadrature method given in [17, 18]. An estimate of the order of magnitude of the errors in the stress intensity factors is attempted by evaluation of the expansion coefficients of the interpolation polynomials as described in [19].

2. GENERAL FORMULAS

The problem under consideration is illustrated in Fig. 1. It consists of an elastic strip of width H containing a linear crack. The crack's inclination with the y-axis is designated ω $(-\pi/2 < \omega < \pi/2)$. In this section the calculations are performed for an internal crack in a strip of finite width. As limiting cases we can obtain solutions for an edge crack and a crack in a half-plane. Details concerning these cases may be found in [20]. The calculations make use of two rectangular coordinate systems, $\{x, y\}$ and $\{n, s\}$. The connection between these systems is given

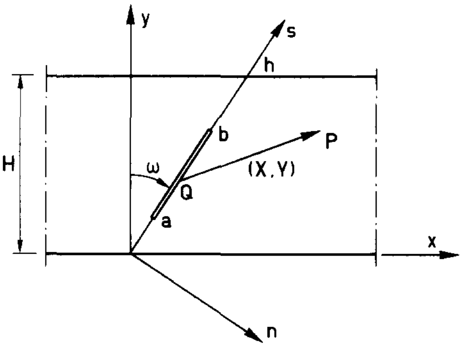

Fig. 1. Coordinate systems and geometrical parameters.

by the orthogonal transformation formula.

$$\begin{Bmatrix} n \\ s \end{Bmatrix} = \begin{Bmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{Bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix} \tag{2.1}$$

In the $\{n, s\}$ system the crack extends from $(0, a)$ to $(0, b)$. The strip is defined in the $\{x, y\}$ system by $0 \le y \le H$.

2.1 *The stress functions*

The strip boundaries are stress free, while normal and shear stresses are prescribed on the crack surfaces. The analysis is limited to the case where the opposite crack surfaces have equal but opposite stress vectors. By simple superposition we can then obtain solutions for the cracked strip with any kind of loading which does not include nonequilibrated loading on the crack provided we can find the corresponding solution for the uncracked strip. Using the $\{n, s\}$ system on the crack surface and the $\{x, y\}$ system on the strip boundaries we have

$$\begin{aligned} \sigma_{nn}(0+, s) = \sigma_{nn}(0-, s) = p(s) \quad a < s < b \\ \sigma_{ns}(0+, s) = \sigma_{ns}(0-, s) = q(s) \quad a < s < b \end{aligned} \tag{2.2a,b}$$

$$\begin{aligned} \sigma_{yy}(x, 0) = \sigma_{yy}(x, H) = 0 \quad -\infty < x < \infty \\ \sigma_{xy}(x, 0) = \sigma_{xy}(x, H) = 0 \quad -\infty < x < \infty \end{aligned} \tag{2.3a,b}$$

The stresses will be described by a stress function Φ defined by

$$\sigma_{nn} = \frac{\partial^2\Phi}{\partial s^2}, \quad \sigma_{ss} = \frac{\partial^2\Phi}{\partial n^2}, \quad \sigma_{ns} = -\frac{\partial^2\Phi}{\partial n \partial s} \tag{2.4a-c}$$

Similar relations hold in the $\{x, y\}$ system. The stress function Φ is constructed from two parts $\Phi^1(n, s)$ and $\Phi^2(x, y)$.

$$\Phi = \Phi^1(n, s) + \Phi^2(x, y) \tag{2.5}$$

This is a generalization of the technique used in [3] for the determination of stresses in a strip with a symmetrically loaded transverse central crack.

$\Phi^1(n, s)$ is the stress function for the full plane with a crack from $(0, a)$ to $(0, b)$ loaded as described by (2.2). Use of complex Fourier transforms immediately yields[21]

$$\Phi^1(n,s)=\begin{cases}\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}[C_1(\eta)+n|n|C_2(\eta)]\,\mathrm{e}^{-|\eta|n}\,\mathrm{e}^{-i\eta s}\,\mathrm{d}\eta, & n>0\\[2ex] \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty}[C_3(\eta)+n|\eta|C_4(\eta)]\,\mathrm{e}^{|\eta|n}\,\mathrm{e}^{-i\eta s}\,\mathrm{d}\eta, & n<0\end{cases} \tag{2.6}$$

In (2.6) use has not yet been made of the fact that the stress vectors on the surfaces $n = 0+$ and $n = 0-$ are of equal magnitude. This is conveniently expressed by

$$\Phi^1(0+, s) = \Phi^1(0-, s)$$
$$\frac{\partial}{\partial n}\Phi^1(0+, s) = \frac{\partial}{\partial n}\Phi^1(0-, s) \tag{2.7a, b}$$

By substituting (2.6) in (2.7) we see that $\Phi^1(n, s)$ is described by two unknown functions. Renaming these $B_1(\eta)$ and $B_2(\eta)$ we get

$$\Phi^1(n,s)=\frac{1}{2\pi}\int_{-\infty}^{\infty}[B_1(\eta)+|\eta||n|B_1(\eta)+|\eta|nB_2(\eta)]\,\mathrm{e}^{-|\eta||n|}\,\mathrm{e}^{-i\eta s}\,\mathrm{d}\eta \tag{2.8}$$

In general $B_1(\eta)$ and $B_2(\eta)$ are complex functions assuming values different from zero in infinite intervals. This is inconvenient for numerical calculations and we obtain unknown functions of finite support in the following way. Using the Lamé constants λ and μ Hooke's law for plane strain is

$$\sigma_{nn} = (\lambda + 2\mu)\frac{\partial u}{\partial n} + \lambda\frac{\partial v}{\partial s}$$

$$\sigma_{ss} = \lambda\frac{\partial u}{\partial n} + (\lambda + 2\mu)\frac{\partial v}{\partial s}$$

$$\sigma_{ns} = \mu\left(\frac{\partial u}{\partial s} + \frac{\partial v}{\partial n}\right) \tag{2.9a–c}$$

u and v are the displacements in the $\{n, s\}$ system. By suitable elimination in (2.9) we obtain

$$\frac{8\mu}{\kappa+1}\frac{\partial}{\partial s}[v(0+,s)-v(0-,s)] = \frac{\partial^2}{\partial n^2}\Phi^1(0+,s) - \frac{\partial^2}{\partial n^2}\Phi^1(0-,s)$$
$$\frac{8\mu}{\kappa+1}\frac{\partial^2}{\partial s^2}[u(0+,s)-u(0-,s)] = -\frac{\partial^3}{\partial n^3}\Phi^1(0+,s) + \frac{\partial^3}{\partial n^3}\Phi^1(0-,s) \tag{2.10a, b}$$

where κ for plane strain is defined by Poisson's ratio ν as $3-4\nu$. The usual modification of λ for plane stress leads to $\kappa = (3-\nu)/(1+\nu)$. We now define the functions $f(s)$ and $g(s)$ by

$$f(s) = \frac{2\mu}{\kappa+1}\frac{\partial}{\partial s}[u(0+,s)-u(0-,s)]$$

$$g(s)=\frac{2\mu}{\kappa+1}\frac{\partial}{\partial s}[v(0+,s)-v(0-,s)] \tag{2.11a, b}$$

As the crack extends from a to b we obviously have

$$f(s)=g(s)=0 \quad \text{for} \quad s<a \quad \text{or} \quad b<s \tag{2.12}$$

Substitution of (2.11) and (2.8) in (2.10) and use of the inverse Fourier transform lead to

$$\eta|\eta|B_1(\eta)=i\int_a^b f(t)\,\mathrm{e}^{i\eta t}\,\mathrm{d}t$$

$$\eta^2 B_2(\eta)=-\int_a^b g(t)\,\mathrm{e}^{i\eta t}\,\mathrm{d}t \tag{2.13a, b}$$

We denote stresses corresponding to $\Phi^1(n,s)$ by index 1. These are found by differentiation of (2.8) followed by substitution of (2.13). The integration with respect to η is carried out by use of the sine and cosine transform formulas listed in Appendix A.

$$\sigma^1_{nn}(n,s)=\frac{\partial^2\Phi^1}{\partial s^2}=\frac{1}{\pi}\int_a^b\left\{\left[\frac{2(t-s)}{n^2+(t-s)^2}+(t-s)\frac{n^2-(t-s)^2}{[n^2+(t-s)^2]^2}\right]f(t)\right.$$
$$\left.+n\frac{n^2-(t-s)^2}{[n^2+(t-s)^2]^2}g(t)\right\}\mathrm{d}t$$

$$\sigma^1_{ss}(n,s)=\frac{\partial^2\Phi^1}{\partial n^2}=\frac{1}{\pi}\int_a^b\left\{-(t-s)\frac{n^2-(t-s)^2}{[n^2+(t-s)^2]^2}f(t)\right.$$
$$\left.+\left[\frac{2n}{n^2+(t-s)^2}-n\frac{n^2-(t-s)^2}{[n^2+(t-s)^2]^2}\right]g(t)\right\}\mathrm{d}t$$

$$\sigma^1_{ns}(n,s)=-\frac{\partial^2\Phi^1}{\partial n\partial s}=\frac{1}{\pi}\int_a^b\left\{n\frac{n^2-(t-s)^2}{[n^2+(t-s)^2]^2}f(t)\right.$$
$$\left.-(t-s)\frac{n^2-(t-s)^2}{[n^2+(t-s)^2]^2}g(t)\right\}\mathrm{d}t \tag{2.14a–c}$$

The stress function $\Phi^2(x,y)$ is the solution for the uncracked strip with boundary loading. Use of the complex Fourier transform yields

$$\Phi^2(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\{[A_1(\xi)+\xi yA_2(\xi)]\,\mathrm{e}^{-\xi y}+[A_3(\xi)+\xi yA_4(\xi)]\,\mathrm{e}^{\xi y}\}\,\mathrm{e}^{-i\xi x}\,\mathrm{d}\xi \tag{2.15}$$

The task now is to determine the four arbitrary functions $A_i(\xi)$ in terms of $f(t)$ and $g(t)$ by means of the strip boundary conditions (2.3).

2.2 *The strip boundary conditions*

The boundary conditions on the strip amount to the requirement that the resultant normal and shear stresses on the boundaries $y = 0$ and $y = H$ must vanish. We introduce the notation

$$\alpha = \cos\omega, \quad \beta = \sin\omega \tag{2.16}$$

and obtain from (2.3), (2.5) and (2.1)

$$\begin{aligned}
\frac{\partial^2\Phi^2}{\partial x^2} &= \alpha^2\frac{\partial^2\Phi^1}{\partial n^2} + \beta^2\frac{\partial^2\Phi^1}{\partial s^2} + 2\alpha\beta\frac{\partial^2\Phi^1}{\partial n\partial s},\\
\frac{\partial^2\Phi^2}{\partial x\partial y} &= \alpha\beta\left(\frac{\partial^2\Phi^1}{\partial s^2} - \frac{\partial^2\Phi^1}{\partial n^2}\right) + (\alpha^2 - \beta^2)\frac{\partial^2\Phi^1}{\partial n\partial s}, \quad y = 0, H
\end{aligned} \tag{2.17a, b}$$

The right-hand sides are expressed in terms of $f(t)$, $g(t)$ and the coordinates $(n, s - t)$ by (2.14). To obtain equations for the unknown functions A_i, $i = 1, \ldots, 4$, it is necessary to apply a complex Fourier transform with respect to the variable x. In order to do that we make the following observation. The vector QP in Fig. 1 has the coordinates $(n, s - t)$ in the $\{n, s\}$ system. According to (2.1) its coordinates (X, Y) in the $\{x, y\}$ system will then satisfy

$$\begin{Bmatrix} n \\ s - t \end{Bmatrix} = \begin{Bmatrix} \alpha & -\beta \\ \beta & \alpha \end{Bmatrix}\begin{Bmatrix} X \\ Y \end{Bmatrix} \tag{2.18}$$

In terms of the coordinates (x, y) to the point P this amounts to

$$X = x - \beta t, \quad Y = y - \alpha t \tag{2.19a, b}$$

We now substitute X and Y in (2.14), and the stress functions from (2.14) and (2.15) are inserted in (2.17). Because of (2.19), a Fourier transform using $e^{i\xi x}$ is easily altered to a Fourier transform using $e^{i\xi X}$. The calculation of the transforms of the right-hand sides of (2.17) is quite extensive and will be omitted here. Use is made of the complex Fourier transforms listed in Appendix A. The result may be written in the form

$$\begin{aligned}
&A_1(\xi) + A_3(\xi) = G_1(\xi)\\
&A_1(\xi) - A_2(\xi) - A_3(\xi) - A_4(\xi) = G_2(\xi)\\
&A_1(\xi)\,e^{-\xi y} + \xi H A_2(\xi)\,e^{-\xi H} + A_3(\xi)\,e^{\xi H} + \xi H A_4(\xi)\,e^{\xi H} = G_3(\xi)\\
&A_1(\xi)\,e^{-\xi y} - (1 - \xi H)A_2(\xi)\,e^{-\xi H} - A_3(\xi)\,e^{\xi H} - (1 + \xi H)A_4(\xi)\,e^{\xi H} = G_4(\xi)
\end{aligned} \tag{2.20a–d}$$

The functions $G_i(\xi)$, $i = 1, \ldots, 4$ do not enter directly into the solution but only through the following combinations

$$\begin{aligned}
H_1(\xi) &= G_1(\xi) + G_2(\xi)\\
H_2(\xi) &= G_1(\xi) - G_2(\xi)\\
H_3(\xi) &= G_3(\xi) + G_4(\xi)\\
H_4(\xi) &= G_3(\xi) - G_4(\xi)
\end{aligned} \tag{2.21a–d}$$

The functions $H_i(\xi)$, $i = 1, \ldots, 4$ are listed in Appendix B. By means of these functions the solution to (2.20) can be expressed in terms of $f(t)$ and $g(t)$.

$$2A_1(\xi) = A_2(\xi) + A_4(\xi) + H_1(\xi)$$
$$2A_3(\xi) = -A_2(\xi) - A_4(\xi) + H_2(\xi) \qquad (2.22a, b)$$

$$A_2(\xi) = \{2\xi H[H_1(\xi) - e^{\xi H}H_3(\xi)] + (e^{\xi H} - e^{-\xi H})[e^{\xi H}H_2(\xi) - H_4(\xi)]\}/[(e^{\xi H} - e^{-\xi H})^2 - 4\xi^2H^2]$$
$$A_4(\xi) = \{2\xi H[H_2(\xi) - e^{-\xi H}H_4(\xi)] + (e^{\xi H} - e^{-\xi H})[e^{-\xi H}H_1(\xi) - H_3(\xi)]\}/[(e^{\xi H} - e^{-\xi H})^2 - 4\xi^2H^2] \qquad (2.23a, b)$$

2.3 *Stress Influence Functions*

Influence functions for the stresses connected with the stress function $\Phi^1(n, s)$ were given in (2.14). The following stress combinations are conveniently used in the evaluation of stresses corresponding to $\Phi^2(x, y)$.

$$[\sigma^2_{xx}(x, y) + \sigma^2_{yy}(x, y)]/2 = \left[\frac{\partial^2\Phi^2}{\partial y^2} + \frac{\partial^2\Phi^2}{\partial x^2}\right]\Big/2 = -\frac{1}{2\pi}\int_{-\infty}^{\infty}\{A_2(\xi)\,e^{-\xi y} - A_4(\xi)\,e^{\xi y}\}\xi^2\,e^{-i\xi x}\,d\xi$$

$$[\sigma^2_{xx}(x, y) - \sigma^2_{yy}(x, y)]/2 = \left[\frac{\partial^2\Phi^2}{\partial y^2} - \frac{\partial^2\Phi^2}{\partial x^2}\right]\Big/2 = \frac{1}{2\pi}\int_{-\infty}^{\infty}\{[A_1(\xi) - A_2(\xi) + \xi yA_2(\xi)]\,e^{-\xi y} + [A_3(\xi) + A_4(\xi) + \xi yA_4(\xi)]\,e^{\xi y}\}\xi^2\,e^{-i\xi x}\,d\xi$$

$$\sigma^2_{xy}(x, y) = -\frac{\partial^2\Phi^2}{\partial x\partial y} = \frac{i}{2\pi}\int_{-\infty}^{\infty}\{-[A_1(\xi) - A_2(\xi) + \xi yA_2(\xi)]\,e^{-\xi y} + [A_3(\xi) + A_4(\xi) + \xi yA_4(\xi)]\,e^{\xi y}\}\xi^2\,e^{-i\xi x}\,d\xi \qquad (2.24a\text{–}c)$$

The interval of integration in (2.24) is divided at $\xi = 0$ and the negative values of ξ are exchanged with $-\xi$. Use of (2.23) and Appendix B then enables us to express the stress combinations (2.24) as integrals of real functions multiplied by $f(t)$ and $g(t)$. We introduce the stress influence functions $S_{ij}(x, y, t)$, $i = 1, 2, 3$, $j = 1, 2$, by the relations

$$[\sigma^2_{xx}(x, y) + \sigma^2_{yy}(x, y)]/2 = \frac{1}{\pi}\int_a^b [S_{11}(x, y, t)f(t) + S_{12}(x, y, t)g(t)]\,dt$$

$$[\sigma^2_{xx}(x, y) - \sigma^2_{yy}(x, y)]/2 = \frac{1}{\pi}\int_a^b [S_{21}(x, y, t)f(t) + S_{22}(x, y, t)g(t)]\,dt$$

$$\sigma_{xy}^2(x, y) = \frac{1}{\pi}\int_a^b [S_{31}(x, y, t)f(t) + S_{32}(x, y, t)g(t)]\,\mathrm{d}t \qquad (2.25a\text{–}c)$$

We define $h = H/\cos\omega$, and all six functions $S_{ij}(x, y, t)$ can now be given in the symmetric form

$$S_{ij}(x, y, t) = \int_0^\infty [M_{ij}(x, y, t, \xi) - M_{ij}(\beta h - x, \alpha h - y, h - t, \xi)]\,\mathrm{d}\xi$$
$$i = 1, 2, 3; \quad j = 1, 2 \qquad (2.26)$$

It is seen from Fig. 1 that this form implies a certain point symmetry with respect to the point $(0, h/2)$ in the $\{n, s\}$ system. The functions $M_{ij}(x, y, t, \xi)$ are listed in Appendix C.

2.4 *The integral equations*

Only the homogeneous part of the boundary conditions (2.2) has been used. The inhomogeneous part consists of prescribed stresses on the crack surface

$$\sigma_{nn}(0, s) = \sigma_{nn}^1(0, s) + \sigma_{nn}^2(0, s) = p(s), \quad a < s < b$$
$$\sigma_{ns}(0, s) = \sigma_{ns}^1(0, s) + \sigma_{ns}^2(0, s) = q(s), \quad a < s < b \qquad (2.27a, b)$$

From the influence functions (2.14) we obtain

$$\sigma_{nn}^1(0, s) = \frac{1}{\pi}\int_a^b \frac{f(t)}{t - s}\,\mathrm{d}t$$
$$\sigma_{ns}^1(0, s) = \frac{1}{\pi}\int_a^b \frac{g(t)}{t - s}\,\mathrm{d}t \qquad (2.28a, b)$$

where the integrals are defined by their Cauchy principal values. $\sigma_{nn}^2(0, s)$ and $\sigma_{ns}^2(0, s)$ are found by transforming the stresses given by (2.25) to the $\{n, s\}$ system. The boundary conditions (2.33) then take the form of two singular integral equations of the first kind.

$$\int_a^b \frac{f(t)}{t - s}\,\mathrm{d}t + \int_a^b K_{11}(s, t)f(t)\,\mathrm{d}t + \int_a^b K_{12}(s, t)g(t)\,\mathrm{d}t = \pi p(s)$$
$$a < s < b$$
$$\int_a^b \frac{g(t)}{t - s}\,\mathrm{d}t + \int_a^b K_{21}(s, t)f(t)\,\mathrm{d}t + \int_a^b K_{22}(s, t)g(t)\,\mathrm{d}t = \pi q(s)$$
$$a < s < b \qquad (2.29a, b)$$

Here we have defined

$$K_{1j}(s, t) = S_{1j}(\beta s, \alpha s, t) + S_{2j}(\beta s, \alpha s, t)\cos(2\omega) - S_{3j}(\beta s, \alpha s, t)\sin(2\omega)$$
$$K_{2j}(s, t) = S_{2j}(\beta s, \alpha s, t)\sin(2\omega) + S_{3j}(\beta s, \alpha s, t)\cos(2\omega)$$
$$j = 1, 2 \qquad (2.30a, b)$$

To retain the continuity of the material outside the crack we must impose the two conditions

$$\int_a^b f(t)\,\mathrm{d}t = 0$$

$$\int_a^b g(t)\,\mathrm{d}t = 0 \tag{2.31a, b}$$

For $0 < a < b < h$ the kernels $K_{ij}(s, t)$ are continuous bounded functions and $f(t)$ and $g(t)$ have the fundamental function $w(t) = (b-t)^{-1/2}(t-a)^{-1/2}$. It is noted that the equations (2.29) are uncoupled for $\omega = 0$.

The formulation presented here allows the determination of the stresses at any point of the cracked strip once $f(t)$ and $g(t)$ have been determined. This is done by evaluating σ^1_{nn}, σ^1_{ss} and σ^1_{ns} by (2.14) and σ^2_{xx}, σ^2_{yy} and σ^2_{xy} by (2.25). The total stresses at the point are obtained by expressing all components in one coordinate system. It is easily realized that an extensive survey of the stresses is a considerable numerical task because the influence functions $S_{ij}(x, y, t)$ must be reevaluated by numerical integration for each new point (x, y).

3. NUMERICAL METHOD

In the case of internal cracks the only singularities of the kernels are of Cauchy type. For this type of integral equations a quadrature method has been established[17]. First the intervals of integration are normalized by introduction of the new variables τ and ζ,

$$t = \tau(b-a)/2 + (b+a)/2$$

$$s = \zeta(b-a)/2 + (b+a)/2 \tag{3.1a, b}$$

In the new formulation we use kernels defined by

$$k_{ij}(\zeta, \tau) = \frac{b-a}{2} K_{ij}(s, t) \tag{3.2}$$

and the unknown functions are

$$\phi(\tau) = F(\tau)(1-\tau^2)^{-1/2} = f(t)$$

$$\psi(\tau) = G(\tau)(1-\tau^2)^{-1/2} = g(t) \tag{3.3a, b}$$

(2.29) and (2.31) then become

$$\int_{-1}^{1} \frac{\phi(\tau)}{\tau-\zeta}\,\mathrm{d}\tau + \int_{-1}^{1} k_{11}(\zeta, \tau)\phi(\tau)\,\mathrm{d}\tau + \int_{-1}^{1} k_{12}(\zeta, \tau)\psi(\tau)\,\mathrm{d}\tau = \pi p(\zeta)$$

$$\int_{-1}^{1} \frac{\psi(\tau)}{\tau-\zeta}\,\mathrm{d}\tau + \int_{-1}^{1} k_{21}(\zeta, \tau)\phi(\tau)\,\mathrm{d}\tau + \int_{-1}^{1} k_{22}(\zeta, \tau)\psi(\tau)\,\mathrm{d}\tau = \pi q(\zeta)$$

$$-1 < \zeta < 1 \tag{3.4a, b}$$

$$\int_{-1}^{1} \phi(\tau)\,\mathrm{d}\tau = 0$$

$$\int_{-1}^{1} \psi(\tau)\,\mathrm{d}\tau = 0 \tag{3.5a, b}$$

In (3.4) $p(\zeta)$ and $q(\zeta)$ are determined from the original definition by (3.1b).

The quadrature method is based on two essential assumptions. It is assumed that $F(\tau)$ and $G(\tau)$ can be approximated by polynomials of finite degree $n-1$. Furthermore it is assumed that the four functions $k_{11}(\zeta, \tau)F(\tau)$, $k_{21}(\zeta, \tau)F(\tau)$, $k_{12}(\zeta, \tau)G(\tau)$ and $k_{22}(\zeta, \tau)G(\tau)$ considered as functions of τ may be approximated by polynomials of degree $2n-1$ [17–19]. Under these assumptions we obtain the following system of linear equations for the determination of $F(\tau_j)$ and $G(\tau_j)$.

$$\sum_{j=1}^{n}\left[\frac{1}{\tau_j-\zeta_i}+k_{11}(\zeta_i,\tau_j)\right]F(\tau_j)+\sum_{j=1}^{n}k_{12}(\zeta_i,\tau_j)G(\tau_j)=np(\zeta_i)$$

$$\sum_{j=1}^{n}k_{21}(\zeta_i,\tau_j)F(\tau_j)+\sum_{j=1}^{n}\left[\frac{1}{\tau_j-\zeta_i}+k_{22}(\zeta_i,\tau_j)\right]G(\tau_j)=nq(\zeta_i) \tag{3.6a, b}$$

$$\sum_{j=1}^{n}F(\tau_j)=0$$

$$\sum_{j=1}^{n}G(\tau_j)=0 \tag{3.7a, b}$$

where we have defined

$$\tau_j=\cos\left[\frac{2j-1}{2n}\pi\right],\quad j=1,2,\ldots,n$$

$$\zeta_i=\cos\left[\frac{i}{n}\pi\right],\qquad i=1,2,\ldots,n-1 \tag{3.8a, b}$$

The kernels $k_{lk}(\zeta_i, \tau_j)$ are evaluated by numerical integration using Filon's formula. In order to check the convergence of the method it is convenient to express the solutions in terms of Chebyshev polynomials of the first kind

$$F(\tau)=c_0/2+\sum_{k=1}^{n-1}c_kT_k(\tau)$$

$$G(\tau)=\mathrm{d}_0/2+\sum_{k=1}^{n-1}\mathrm{d}_kT_k(\tau) \tag{3.9a, b}$$

As proved in [19] the coeffilcients are determined by

$$c_k=\frac{2}{n}\sum_{j=1}^{n}\cos\left[\frac{2j-1}{2n}k\pi\right]F(\tau_j),\qquad k=0,1,\ldots,n-1 \tag{3.10}$$

and the similar formula for d_k.

The quantities of main importance are the stress intensity factors. We define them by

$$k_1(b)=\lim_{s\to b+}\sqrt{2(s-b)}\,\sigma_{nn}(s,0)$$

$$k_2(b) = -\lim_{s\to b+} \sqrt{2(s-b)}\,\sigma_{ns}(s,0) \tag{3.11a, b}$$

and the similar formulas for $s = a$. The minus sign in (3.11b) is introduced because the crack is along the second axis in the $\{n, s\}$ system. The stress intensity factors are proportional to the values of $F(\tau)$ and $G(\tau)$ at the corresponding endpoints.

$$k_1(a) = F(-1)\sqrt{(b-a)/2}, \quad k_2(a) = -G(1)\sqrt{(b-a)/2}$$

$$k_1(b) = -F(1)\sqrt{(b-a)/2}, \quad k_2(b) = G(-1)\sqrt{(b-a)/2} \tag{3.12a, b}$$

The value $F(1)$ may be determined directly as

$$F(1) = \frac{1}{n}\sum_{j=1}^{n} \frac{\sin\left[\frac{2n-1}{4n}(2j-1)\pi\right]}{\sin\left[\frac{2j-1}{4n}\pi\right]} F(\tau_j) \tag{3.13}$$

$F(-1)$, $G(1)$ and $G(-1)$ are determined from similar formulas[19].

4. NUMERICAL RESULTS

The numerical results are given for three load situations. These are chosen as the loads arising from tension, bending and shear according to technical beam theory. In the $\{x, y\}$ system the stresses in the uncracked strip are:

Tension;

$$\sigma_{xx} = \sigma_m, \quad \sigma_{yy} = 0, \quad \sigma_{xy} = 0 \tag{4.1a–c}$$

Bending;

$$\sigma_{xx} = \sigma_m(1 - 2y/H), \quad \sigma_{yy} = 0, \quad \sigma_{xy} = 0 \tag{4.2a–c}$$

Shear;

$$\sigma_{xx} = 0, \quad \sigma_{xy} = 0, \quad \sigma_{xy} = 4\sigma_m(y/H - 1)y/H \tag{4.3a–c}$$

$p(s)$ and $q(s)$ are found from (4.1–4.3) by transforming the stresses to the $\{n, s\}$ system and changing the sign. Again we use $h = H/\cos\omega$.

Tension:

$$p(s) = -\alpha^2\sigma_m, \quad q(s) = -\alpha\beta\sigma_m \tag{4.4a, b}$$

Bending;

$$p(s) = \alpha^2\sigma_m(2s/h - 1), \quad q(s) = \alpha\beta\sigma_m(2s/h - 1) \tag{4.5a, b}$$

Shear;

$$p(s) = 8\alpha\beta\sigma_m(s/h - 1)s/h$$
$$q(s) = -4(\alpha^2 - \beta^2)\sigma_m(s/h - 1)s/h \qquad (4.6a, b)$$

The stress intensity factors are normalized with respect to the maximum stress σ_m and a length parameter

$$l = (b - a)/2 \qquad (4.7)$$

The crack size is characterized by the dimensionless parameter

$$\lambda = (b - a)/H \qquad (4.8)$$

First a central crack with $\omega = 0$ is considered (Table 1). An estimate of the relative error is given in the last column. It has been obtained by means of Chebyshev series expansions and is of empirical nature [19]. The same problem has been considered in [3], [13] and [14] for pure tension (Table 2). There is good agreement and it is seen that for moderate values of λ high accuracy is obtained by use of the empirical formula in the last column. Tables 3 and 4 contain the same

Table 1. Stress intensity factors at a for a central crack, $\omega = 0$

$\omega = 0$ λ	Tension $k_1/\sigma_m\sqrt{l}$	Bending $k_1/\sigma_m\sqrt{l}$	Shear $k_2/\sigma_m\sqrt{l}$	Relative error
0·1	1·0060	0·05000	1·0017	10^{-4}
0·2	1·0246	0·10007	1·0071	10^{-4}
0·3	1·0577	0·15051	1·0175	10^{-4}
0·4	1·1094	0·20225	1·0353	10^{-4}
0·5	1·1867	0·25724	1·0648	10^{-4}
0·6	1·3033	0·31965	1·1148	2.10^{-4}
0·7	1·4884	0·39864	1·2041	2.10^{-4}
0·8	1·8169	0·51851	1·3801	2.10^{-4}
0·9	2·585	0·7777	1·835	5.10^{-4}
0·95	4·252	1·114	2·332	10^{-3}

Table 2. Stress intensity factors for a central crack in a strip in tension, $\omega = 0$

$\omega = 0$ λ	Ref. [14] $k_1/\sigma_m\sqrt{l}$	Ref. [3] $k_1/\sigma_m\sqrt{l}$	$(1/\sqrt{\cos(\lambda\pi/2)})$
0·1	1·0060	1·007	1·0062
0·2	1·0246	1·026	1·0254
0·3	1·0577	1·059	1·0594
0·4	1·1094	1·114	1·1118
0·5	1·1867	1·194	1·1892
0·6	1·3033	1·309	1·3043
0·7	1·4882	1·500	1·4841
0·8	1·8160	1·826	1·7989
0·9	2·58	2·580	2·5283

information for $\omega = \pi/6$ and $\omega = \pi/3$. For small values of λ the stress intensity factors may be estimated by use of Table 1 and simple transform of the crack surface loading. For the case of tension we can obtain the stress intensity factors from curves given in [10] for $\lambda = 1/3$. For $\omega = \pi/6$ we get

$$k_1 = 0{\cdot}806\, \sigma_m \sqrt{l}, \quad k_2 = -0{\cdot}440\, \sigma_m \sqrt{l}$$

and for $\omega = \pi/3$

$$k_1 = 0{\cdot}276\, \sigma_m \sqrt{l}, \quad k_2 = -0{\cdot}455\, \sigma_m \sqrt{l}$$

A certain difference from the values in Tables 3 and 4 is noted. Both here and in the previous case this method is believed to yield superior accuracy due to the efficient numerical treatment of the

Table 3. Stress intensity factors at a for a central crack, $\omega = \pi/6$

$\omega = \pi/6$	Tension		Bending		Shear	
λ	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$
0·1	0·7557	−0·4338	0·0325	−0·0187	0·8713	0·5025
0·2	0·7729	−0·4364	0·0650	−0·0375	0·8876	0·5099
0·3	0·8023	−0·4412	0·0978	−0·0563	0·9160	0·5223
0·4	0·8452	−0·4488	0·1313	−0·0753	0·9589	0·5396
0·5	0·9039	−0·4607	0·1665	−0·0948	1·0200	0·5622
0·6	0·9827	−0·4787	0·2053	−0·1154	1·1057	0·5909
0·7	1·0898	−0·5066	0·2507	−0·1384	1·2274	0·6278
0·8	1·2417	−0·5510	0·3082	−0·1661	1·4063	0·6782
0·9	1·4765	−0·6285	0·3907	−0·2047	1·6915	0·7541

Table 4. Stress intensity factors at a for a central crack, $\omega = \pi/3$

$\omega = \pi/3$	Tension		Bending		Shear	
λ	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$
0·1	0·2527	−0·4352	0·00625	−0·01083	0·8746	−0·5017
0·2	0·2605	−0·4416	0·01251	−0·02166	0·9000	−0·5067
0·3	0·2732	−0·4518	0·01882	−0·03254	0·9414	−0·5149
0·4	0·2900	−0·4654	0·02526	−0·04354	0·9975	−0·5263
0·5	0·3105	−0·4817	0·03197	−0·05479	1·0671	−0·5408
0·6	0·3341	−0·5002	0·03906	−0·06642	1·1494	−0·5584
0·7	0·3606	−0·5209	0·04669	−0·07861	1·2440	−0·5795
0·8	0·3900	−0·5435	0·05501	−0·09154	1·3515	−0·6046
0·9	0·4225	−0·5684	0·06415	−0·10540	1·4733	−0·6342

Table 5. Stress intensity factors at a for an eccentric crack, $\omega = 0$, $\epsilon = 0{\cdot}2$

$\omega = 0$	Tension	Bending	Shear
λ	$k_1/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$
0·1	1·0079	0·2516	0·9420
0·2	1·0340	0·3071	0·9292
0·3	1·0856	0·3687	0·9240
0·4	1·1743	0·4410	0·9313
0·5	1·3264	0·5351	0·9609
0·6	1·6096	0·6800	1·0363
0·7	2·2917	0·9935	1·2422

Table 6. Stress intensity factors at a for an eccentric crack, $\omega = \pi/6$, $\epsilon = 0{\cdot}2$

$\omega = \pi/6$	Tension		bending		Shear	
λ	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$
0·1	0·7569	−0·4344	0·1839	−0·1056	0·8228	0·4739
0·2	0·7785	−0·4397	0·2208	−0·1254	0·8289	0·4726
0·2	0·8171	−0·4509	0·2619	−0·1465	0·8534	0·4751
0·4	0·8768	−0·4709	0·3091	−0·1696	0·9019	0·4806
0·5	0·9656	−0·5047	0·3665	−0·1965	0·9836	0·4875
0·6	1·1006	−0·5619	0·4426	−0·2311	1·1167	0·4942
0·7	1·3255	−0·6657	0·5577	−0·2830	1·3440	0·4977

Table 7. Stress intensity factors at a for an eccentric crack, $\omega = \pi/3$, $\epsilon = 0{\cdot}2$

$\omega = \pi/3$	Tension		Bending		Shear	
λ	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$	$k_1/\sigma_m\sqrt{l}$	$k_2/\sigma_m\sqrt{l}$
0·1	0·2528	−0·4356	0·0568	−0·0979	0·8321	−0·4771
0·2	0·2605	−0·4436	0·0647	−0·1104	0·8508	−0·4796
0·3	0·2721	−0·4569	0·0734	−0·1239	0·8860	−0·4889
0·4	0·2866	−0·4752	0·0832	−0·1386	0·9360	−0·5055
0·5	0·3039	−0·4979	0·0941	−0·1545	0·9996	−0·5296
0·6	0·3239	−0·5245	0·1063	−0·1718	1·0765	−0·5614
0·7	0·3471	−0·5554	0·1201	−0·1909	1·1674	−0·6016

singularities. Tables 5 and 7 contain similar results for eccentric cracks. The eccentricity is characterized by the parameter

$$\epsilon = 1 - (b + a)/h \tag{4.9}$$

Further results have been given in [20].

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APPENDIX A

The transform formulas used are listed in pairs where the first is a simple sine or cosine transform and the second is the corresponding inverse complex Fourier transform.

$$\int_0^\infty e^{-\xi|Y|}\cos(X\xi)\,d\xi = \frac{|Y|}{X^2+Y^2}$$

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{Y}{X^2+Y^2}e^{i\xi X}\,dX = \operatorname{sgn}(Y)\,e^{-|\xi Y|} \tag{A.1}$$

$$\int_0^\infty e^{-\xi|Y|}\sin(X\xi)\,d\xi = \frac{X}{X^2+Y^2}$$

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{X}{X^2+Y^2}e^{i\xi X}\,dX = i\operatorname{sgn}(\xi)\,e^{-|\xi Y|} \tag{A.2}$$

$$\int_0^\infty \xi Y\,e^{-\xi|Y|}\cos(X\xi)\,d\xi = -Y\frac{X^2-Y^2}{[X^2+Y^2]^2}$$

$$\frac{1}{\pi}\int_{-\infty}^\infty Y\frac{X^2-Y^2}{[X^2+Y^2]^2}e^{i\xi X}\,dX = -|\xi|Y\,e^{-|\xi Y|} \tag{A.3}$$

$$\int_0^\infty \xi|Y|\,e^{-\xi|Y|}\sin(X\xi)\,d\xi = \frac{2XY^2}{[X^2+Y^2]^2}$$

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{2XY^2}{[X^2+Y^2]^2}e^{i\xi X}\,dX = i\xi|Y|\,e^{-|\xi Y|} \tag{A.4}$$

$$\int_0^\infty (1-\xi|Y|)\,e^{-\xi|Y|}\cos(X\xi)\,d\xi = \frac{2X^2|Y|}{[X^2+Y^2]^2}$$

$$\frac{1}{\pi}\int_{-\infty}^\infty \frac{2X^2Y}{[X^2+Y^2]^2}e^{i\xi X}\,dX = \operatorname{sgn}(Y)(1-|\xi||Y|)\,e^{-|\xi Y|} \tag{A.5}$$

$$\int_0^\infty (1-\xi|Y|)\,e^{-\xi|Y|}\sin(X\xi)\,d\xi = X\frac{X^2-Y^2}{[X^2+Y^2]^2}$$

$$\frac{1}{\pi}\int_{-\infty}^\infty X\frac{X^2-Y^2}{[X^2+Y^2]^2}e^{i\xi X}\,dX = i\operatorname{sgn}(\xi)(1-|\xi||Y|)\,e^{-|\xi Y|} \tag{A.6}$$

APPENDIX B

The function $H_i(\xi)$, $i=1,\dots,4$ from (2.21–23). Note that ξ is a positive variable in all expressions in this appendix. h is defined as H/α.

$$\xi^2 H_1(\xi)=\int_a^b f(t)\mathrm{e}^{i(\beta\xi t-\omega)}\mathrm{e}^{-\alpha\xi t}\,\mathrm{d}t+i\int_a^b g(t)\mathrm{e}^{i(\beta\xi t-\omega)}\mathrm{e}^{-\alpha\xi t}\,\mathrm{d}t$$

$$\xi^2 H_1(-\xi)=\int_a^b f(t)[-\mathrm{e}^{-i(\beta\xi t+\omega)}+2\alpha\xi t\,\mathrm{e}^{-i(\beta\xi t-\omega)}]\,\mathrm{e}^{-\alpha\xi t}\,\mathrm{d}t$$

$$-i\int_a^b g(t)[\mathrm{e}^{-i(\beta\xi t+\omega)}+2\alpha\xi t\,\mathrm{e}^{-i(\beta\xi t-\omega)}]\,\mathrm{e}^{-\alpha\xi t}\,\mathrm{d}t \tag{B.1}$$

$$\xi^2 H_2(\xi)=\int_a^b f(t)[-\mathrm{e}^{i(\beta\xi t+\omega)}+2\alpha\xi t\,\mathrm{e}^{i(\beta\xi t-\omega)}]\,\mathrm{e}^{-\alpha\xi t}\,\mathrm{d}t$$

$$+i\int_a^b g(t)[\mathrm{e}^{i(\beta\xi t+\omega)}+2\alpha\xi t\,\mathrm{e}^{i(\beta\xi t-\omega)}]\,\mathrm{e}^{-\alpha\xi t}\,\mathrm{d}t$$

$$\xi^2 H_2(-\xi)=\int_a^b f(t)\,\mathrm{e}^{-i(\beta\xi t-\omega)}\,\mathrm{e}^{-\alpha\xi t}\,\mathrm{d}t-i\int_a^b g(t)\,\mathrm{e}^{-i(\beta\xi t-\omega)}\,\mathrm{e}^{-\alpha\xi t}\,\mathrm{d}t \tag{B.2}$$

$$\xi^2 H_3(\xi)=\int_a^b f(t)[\mathrm{e}^{i(\beta\xi t-\omega)}-2\alpha\xi(h-t)\,\mathrm{e}^{i(\beta\xi t+\omega)}]\,\mathrm{e}^{-\alpha\xi(h-t)}\,\mathrm{d}t$$

$$+i\int_a^b g(t)[\mathrm{e}^{i(\beta\xi t-\omega)}+2\alpha\xi(h-t)\,\mathrm{e}^{i(\beta\xi t+\omega)}]\,\mathrm{e}^{-\alpha\xi(h-t)}\,\mathrm{d}t$$

$$\xi^2 H_3(-\xi)=-\int_a^b f(t)\,\mathrm{e}^{-i(\beta\xi t+\omega)}\,\mathrm{e}^{-\alpha\xi(h-t)}\,\mathrm{d}t-i\int_a^b g(t)\,\mathrm{e}^{-i(\beta\xi t+\omega)}\,\mathrm{e}^{-\alpha\xi(h-t)}\,\mathrm{d}t \tag{B.3}$$

$$\xi^2 H_4(\xi)=-\int_a^b f(t)\,\mathrm{e}^{i(\beta\xi t+\omega)}\,\mathrm{e}^{-\alpha\xi(h-t)}\,\mathrm{d}t+i\int_a^b g(t)\,\mathrm{e}^{i(\beta\xi t+\omega)}\,\mathrm{e}^{-\alpha\xi(h-t)}\,\mathrm{d}t$$

$$\xi^2 H_4(-\xi)=\int_a^b f(t)[\mathrm{e}^{-i(\beta\xi t-\omega)}-2\alpha\xi(h-t)\,\mathrm{e}^{-i(\beta\xi t+\omega)}]\,\mathrm{e}^{-\alpha\xi(h-t)}\,\mathrm{d}t$$

$$-i\int_a^b g(t)[\mathrm{e}^{-i(\beta\xi t-\omega)}+2\alpha\xi(h-t)\,\mathrm{e}^{-i(\beta\xi t+\omega)}]\,\mathrm{e}^{-\alpha\xi(h-t)}\,\mathrm{d}t \tag{B.4}$$

APPENDIX C

The functions $M_{ij}(x,y,t,\xi)$ from (2.26).

$$M_{11}(x,y,t,\xi)=$$

$$\{[(\mathrm{e}^{2\xi H}-1)\cos(\xi(\beta t-x)+\omega)-(2\xi H+2\alpha t\xi(\mathrm{e}^{2\xi H}-1))\cos(\xi(\beta t-x)-\omega)]\,\mathrm{e}^{-\xi(\alpha t+y)}$$
$$-[2\xi H\cos(\xi(\beta t-x)+\omega)-(1-\mathrm{e}^{-2\xi H}+4\alpha tH\xi^2)\cos(\xi(\beta t-x)-\omega)]$$
$$\mathrm{e}^{-\xi(\alpha t-y)}\}/[(\mathrm{e}^{\xi H}-\mathrm{e}^{-\xi H})^2-4\xi^2H^2] \tag{C.1}$$

$$M_{12}(x,y,t,\xi)=$$

$$\{[(\mathrm{e}^{2\xi H}-1)\sin(\xi(\beta t-x)+\omega)+(2\xi H+2\alpha t\xi(\mathrm{e}^{2\xi H}-1))\sin(\xi(\beta t-x)-\omega)]\,\mathrm{e}^{-\xi(\alpha t+y)}$$

$$-[2\xi H \sin(\xi(\beta t - x) + \omega) + (1 - e^{-2\xi H} + 4\alpha t H\xi^2)\sin(\xi(\beta t - x) - \omega)]$$
$$e^{-\xi(\alpha t - y)}\}/[(e^{\xi H} - e^{-\xi H})^2 - 4\xi^2 H^2] \qquad \text{(C.2)}$$

$M_{21}(x, y, t, \xi) =$

$$\tfrac{1}{2}\{[(e^{2\xi H} - 1)\cos(\xi(\beta t - x) + \omega) - (2\xi H + 2\alpha t\xi(e^{2\xi H} - 1))\cos(\xi(\beta t - x) - \omega)]$$
$$\cdot [1 + 2\xi(H - y) + e^{-2\xi(H-y)}]\, e^{-\xi(\alpha t + y)}$$
$$-[2\xi H \cos(\xi(\beta t - x) + \omega) - (1 - e^{-2\xi H} + 4\alpha t H\xi^2)\cos(\xi(\beta t - x) - \omega)]$$
$$\cdot [1 - 2\xi(H - y) + e^{2\xi(H-y)}]\, e^{-\xi(\alpha t - y)}\}/[(e^{\xi H} - e^{-\xi H})^2 - 4\xi^2 H^2] \qquad \text{(C.3)}$$

$M_{22}(x, y, t, \xi) =$

$$\tfrac{1}{2}\{[(e^{2\xi H} - 1)\sin(\xi(\beta t - x) + \omega) + (2\xi H + 2\alpha t\xi(e^{2\xi H} - 1))\sin(\xi(\beta t - x) - \omega)]$$
$$\cdot [1 + 2\xi(H - y) + e^{-2\xi(H-y)}]\, e^{-\xi(\alpha t + y)}$$
$$-[2\xi H \sin(\xi(\beta t - x) + \omega) + (1 - e^{-2\xi H} + 4\alpha t H\xi^2)\sin(\xi(\beta t - x) - \omega)]$$
$$\cdot [1 - 2\xi(H - y) + e^{2\xi(H-y)}]\, e^{-\xi(\alpha t - y)}\}/[(e^{\xi H} - e^{-\xi H})^2 - 4\xi^2 H^2] \qquad \text{(C.4)}$$

$M_{31}(x, y, t, \xi) =$

$$\tfrac{1}{2}\{[(e^{2\xi H} - 1)\sin(\xi(\beta t - x) + \omega) - (2\xi H + 2\alpha t\xi(e^{2\xi H} - 1))\sin(\xi(\beta t - x) - \omega)]$$
$$\cdot [1 + 2\xi(H - y) - e^{-2\xi(H-y)}]\, e^{-\xi(\alpha t + y)}$$
$$+[2\xi H \sin(\xi(\beta t - x) + \omega) - (1 - e^{-2\xi H} + 4\alpha t H\xi^2)\sin(\xi(\beta t - x) - \omega)]$$
$$\cdot [1 - 2\xi(H - y) - e^{2\xi(H-y)}]\, e^{-\xi(\alpha t - y)}\}/[(e^{\xi H} - e^{-\xi H})^2 - 4\xi^2 H^2] \qquad \text{(C.5)}$$

$M_{32}(x, y, t, \xi) =$

$$-\tfrac{1}{2}\{[(e^{2\xi H} - 1)\cos(\xi(\beta t - x) + \omega) + (2\xi H + 2\alpha t\xi(e^{2\xi H} - 1))\cos(\xi(\beta t - x) - \omega)]$$
$$\cdot [1 + 2\xi(H - y) - e^{-2\xi(H-y)}]\, e^{-\xi(\alpha t + y)}$$
$$+[2\xi H \cos(\xi(\beta t - x) + \omega) + (1 - e^{-2\xi H} + 4\alpha t H\xi^2)\cos(\xi(\beta t - x) - \omega)]$$
$$\cdot [1 - 2\xi(H - y) - e^{2\xi(H-y)}]\, e^{-\xi(\alpha t - y)}\}/[(e^{\xi H} - e^{-\xi H})^2 - 4\xi^2 H^2] \qquad \text{(C.6)}$$